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An Observer for Infinite-Dimensional Dissipative Bilinear Systems

C.-Z. Xu
INRIA-Lorraine (Projet CONGE) & URA CNRS 399 (MMAS)
CESCOM 4, rue Marconi, 57070 – Metz, France
xu@ilm.loria.fr

P. Ligarius and J.-P. Gauthier
INSA-LMI de Rouen, URA CNRS 1378, Department of Mathematics
Place émile Blondel, 76131, Mont-st-Aignan Cedex, France
gauthier@lmi.insa-rouen.fr

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Abstract—We consider bilinear systems of the form

$$\dot{x}(t) = Ax(t) + u(t)Bx(t), \quad y(t) = \langle x(t), c\rangle$$

on an infinite-dimensional Hilbert space H, where A is the generator of a semigroup of contraction, B is a bounded dissipative operator and $c \in H$. The input signal $u \in L^\infty(\mathbb{R}^+)$ such that $u(t) \geq 0$ for almost every $t \in \mathbb{R}^+$. We present a simple observer for this class of systems with the estimation error converging weakly to zero in H for every sufficiently rich input (inputs that we call "regularly persistent"). Our result is a generalization of the previous results in [1,2].

Keywords—Observers, Infinite-dimensional systems, Weak and strong convergence.

1. INTRODUCTION AND THE MAIN RESULT

In our paper [2] and previously in [1], we have proposed a simple observer for infinite dimensional skew-adjoint bilinear systems. This observer has already been studied in the linear time-invariant case by several authors ([3], for instance).

The main result in this paper is that, under suitable observability assumptions, our technique also works for bilinear dissipative (instead of only skew-adjoint) systems. Unfortunately, the proof in [2] for the skew-adjoint case, heavily depends on the fact that skew adjoint linear operators, generate a one parameter group of unitary operators. It turns out that different techniques from those in [2] have to be used in order to prove error convergence of the observer in the dissipative case. One of the important points is a certain result of continuity of the solutions, which is well known for finite dimensional control-affine systems.

We will consider a separable Hilbert space H (with inner product $\langle \cdot, \cdot\rangle$), and dissipative bilinear system of the form:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + u(t)Bx(t), \\ &y(t)\,\langle x(t), c\rangle, \qquad x(0) = x_0 \in H, \end{aligned} \tag{1}$$

where A is the generator of a contraction semigroup, B is a bounded dissipative operator in H and $c \in H$. We assume throughout the paper that the input $u \in L^\infty(\mathbb{R}^+)$ and is such that $u(t) \geq 0$ for almost every $t \in \mathbb{R}^+$. Such an input signal u will be called a positive element

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX of $L^\infty(\mathbb{R}^+)$. The candidate observer system of (1) is given by the following evolution equation on H:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + u(t)B\hat{x}(t) - c(\hat{y}(t) - y(t)),\\ \hat{y}(t) = \langle \hat{x}(t), c\rangle, \qquad \hat{x}(0) = \hat{x}_0 \in H. \tag{2}$$

We will prove that for all positive regularly persistent input $u(t)$ (a notion precisely defined later on, roughly speaking, an input that ensures a minimum amount of observability when the time is passing by), the observer error $\varepsilon(t) = \hat{x}(t) - x(t)$ converges weakly to zero as $t \to +\infty$. If any positive element $u \in L^\infty(\mathbb{R}^+)$ is a T-universal input for some fixed $T > 0$, then the estimation error converges weakly to zero for any positive element of $u \in L^\infty(\mathbb{R}^+)$. We should point out that, if the bounded operator B is skew-adjoint, there is no need to impose that the input signal $u(t) \geq 0$ for almost every $t \in \mathbb{R}^+$. This case has been studied in [2]. In particular, the proof presented here is also true for this case. If the resolvent of the generator A is compact and the input signal converges in the L^1-norm to a constant T-universal input, we will also guarantee strong convergence of the observer error (see [2]). Finally, the extension of our result to nonhomogeneous bilinear systems is obvious.

A bilinear system (the input being known) is, from the point of view of observation and reconstruction of the states, nothing but a linear time dependent system (see [4] for finite dimensional systems). We will define the associated Gram-observability operator. In the paper, all the solutions of the evolution equations are taken in the sense of mild solutions. For instance, a solution of (1) is the unique function $x(\cdot) \in C([s,T];H)$ satisfying the integral equation

$$x(t) = e^{(t-s)A}x_0 + \int_s^t e^{(t-\xi)A}u(\xi)Bx(\xi)\,d\xi \tag{3}$$

with $x_0 \in H$ and $0 \leq s \leq t \leq T$. This integral equation defines uniquely a family of mild evolution operators denoted by $\Phi_u(t,s)$ (see [5, p. 40], or [6, p. 135]). We can write $x(t) = \Phi_u(t,s)x_0$. The generator of a C_0-semigroup perturbed by a bounded linear operator is still the generator of a C_0-semigroup. Also, there is a unique family of mild evolution operators $\Psi_u(t,s)$ associated with the generator

$$A - c\langle\cdot, c\rangle + u(t)B.$$

Hence, the estimation error equation

$$\dot{\varepsilon}(t) = A\varepsilon(t) + u(t)B\varepsilon(t) - c\langle\varepsilon(t), c\rangle,\\ \varepsilon(s) = \varepsilon_0 \in H \tag{4}$$

is well-posed on H as well as the observer equation (2). We can write the estimation error $\varepsilon(t) = \Psi_u(t,s)\varepsilon_0$. For each $T > 0$ and $u \in L^\infty[0,T]$, the Gram-observability operator $W(u,T)$ is a compact operator on H: For any $x \in H$,

$$W(u,T)x = \int_0^T \langle x, \Phi_u^*(t,0)c\rangle\, \Phi_u^*(t,0)c\,dt, \tag{5}$$

where $\Phi_u^*(t,0)$ denotes the adjoint operator of $\Phi_u(t,0)$. For finite-dimensional bilinear systems, if an input $u(t)$ is regularly persistent in the sense of [2], then the associated Gram-observability matrix is uniformly bounded from below. However, for infinite-dimensional systems of the form (1), the Gram-observability operator is compact and so cannot be bounded from below.

We have now to introduce the following definitions.

DEFINITION 1. *An input signal $u \in L^\infty(R^+)$ is called T-universal for some $T > 0$ (depending on u) if $W(u,T)x = 0$ implies that $x = 0$.*

It is well known that a bounded set of $L^\infty[0,T]$ is precompact with respect to the weak* topology. In other words, given a bounded sequence u_n, $\|u_n\|_{L^\infty[0,T]} \le M$, we can extract a subsequence u_{n_k} so that there exists a unique $u^* \in L^\infty[0,T]$ such that for each $f \in L^1[0,T]$

$$\lim_{k\to+\infty} \int_0^T [u_{n_k}(t) - u^*(t)]\, f(t)\, dt = 0.$$

It is also equivalent to say that for each $f \in L^1([0,T];H)$

$$\lim_{k\to+\infty} \left\| \int_0^T [u_{n_k}(t) - u^*(t)]\, f(t)\, dt \right\|_H = 0.$$

(See [2] for a proof of this equivalence).

For each $u \in L^\infty(\mathbb{R}^+)$, and $\theta > 0$, we set $u_{[\theta]}(t) = u(\theta + t)$, the θ-translated input function.

DEFINITION 2. *An input $u \in L^\infty(\mathbb{R}^+)$ is a regularly persistent input for (1) if there exist a time interval $T > 0$ and a real sequence θ_n tending to $+\infty$ as $n \to +\infty$, and the difference $\theta_{n+1} - \theta_n$ being bounded such that the translated input $u_{[\theta_n]}$ converges to a T-universal signal u^* in the weak* topology.*

This means, as in the finite-dimensional case, that $u_{[\theta_n]}(\cdot)$ tends to make the system observable in the same way as u^*. Since $\theta_{n+1} - \theta_n$ is bounded, it means also that the size of the intervals on which u makes the system unobservable does not increase. In particular, every T-periodic input function $u(t)$ which is T-universal is regularly persistent.

Then our main result is the following.

THEOREM 1. *For a regularly persistent positive input $u \in L^\infty(\mathbb{R}^+)$, the estimation error $\varepsilon(t)$ converges weakly to zero in H as t goes to $+\infty$.*

An immediate and important consequence of this theorem is the following.

COROLLARY 1. *If every positive $u \in L^\infty(\mathbb{R}^+)$ is a T-universal input for some fixed $T > 0$ (independent of u), then the observer error $\varepsilon(t)$ converges weakly to zero in H for any positive $u \in L^\infty(\mathbb{R}^+)$.*

Hence, as the reader may remark, we have obtained weak convergence of the estimation error under the same assumption as that for finite dimensional bilinear systems. In our case, the error convergence is weak because of the infinite dimensional structure on H. A natural twofold question is:

(1) Under which condition on the bilinear system (1) could we have stronger convergence of the estimation error (i.e., strong or exponential convergence)?
(2) If the output observation is boundary, that is, the output element c, is not necessarily in the Hilbert space H (cf., [7] for a proper formulation), is it possible to prove stronger convergence of the estimation error?

These might be interesting open questions for our future research. An application is being pursued in this direction based on the results of [8].

The next section will be devoted to the proof of the main result via several technical lemmas.

2. PROOF OF THE MAIN RESULTS

LEMMA 1. *For each $T > 0$, each fixed s with $0 \le s \le T$ and each $x \in H$, we define the map*

$$\Lambda_{s,x} : L^\infty[s,T] \longrightarrow C([s,T];H)$$

such that

$$\Lambda_{s,x}(u)(t) = \Phi_u(t,s)x.$$

Then the map $\Lambda_{s,x}$ is continuous with respect to the weak topology on $L^\infty[s,T]$ and the uniform topology on $C([s,T];H)$.*

PROOF. As stated in the introduction, by definition, we have

$$\Phi_u(t,s)x = e^{(t-s)A}x + \int_s^t e^{(t-\tau)A}u(\tau)B\Phi_u(\tau,s)x\,d\tau. \tag{6}$$

Since the semigroup e^{tA} is contractive, it follows from the Gronwall's Lemma that

$$\sup_{t\in[s,T]} \|\Phi_u(t,s)x\|_H \le \|x\| \exp\left[T\|u\|_{L^\infty}\|B\|\right] \le \tilde{M}.$$

It is clear that the function $\Phi_u(t,s)x$ is continuous with respect to t. Consider a weak* convergent sequence $u_n \in L^\infty[s,T]$ which is bounded. Then,

$$\Phi_{u_n}(t,s)x - \Phi_u(t,s)x = \int_s^t e^{(t-\tau)A}u_n(\tau)B\Phi_{u_n}(\tau,s)x\,d\tau - \int_s^t e^{(t-\tau)A}u(\tau)B\Phi_u(\tau,s)x\,d\tau. \tag{7}$$

Therefore, it is immediate that

$$\begin{aligned}\|\Phi_{u_n}(t,s)x - \Phi_u(t,s)x\| \le &\left\|\int_s^t [u_n(\tau)-u(\tau)]\,e^{(t-\tau)A}B\Phi_u(\tau,s)x\,d\tau\right\| \\ &+ \int_s^t \|u_n\|_{L^\infty}\|B\|\,\|\Phi_{u_n}(\tau,s)x - \Phi_u(\tau,s)x\|\,d\tau. \end{aligned}\tag{8}$$

Now let us prove that, for any $\epsilon > 0$, there is a large integer N such that for all $n \ge N$ and all $t \in [s,T]$,

$$\left\|\int_s^t [u_n(\tau)-u(\tau)]\,e^{(t-\tau)A}B\Phi_u(\tau,s)x\,d\tau\right\| < \epsilon.$$

Take a sequence $u_n \in L^\infty$ weakly* converging to u such that $\|u_n\|_{L^\infty} \le M$. Given any $\epsilon > 0$, we can divide $[s,T]$ into a finite number of intervals $[t_i,t_{i+1}]$ with $i = 0,1,\ldots,m-1$ such that for each i,

$$(t_{i+1}-t_i)\,\|B\|\left(M + \|u\|_{L^\infty}\right)\tilde{M} < \frac{\epsilon}{2}.$$

Since $t \in [s,T]$, t belongs to one of the above intervals: $t \in [t_l,t_{l+1}]$. Then,

$$\begin{aligned}&\left\|\int_s^t [u_n(\tau)-u(\tau)]\,e^{(t-\tau)A}B\Phi_u(\tau,s)x\,d\tau\right\|_H \\ &\le \left\|\int_s^{t_l} [u_n(\tau)-u(\tau)]\,e^{(t-\tau)A}B\Phi_u(\tau,s)x\,d\tau\right\|_H + (t_{l+1}-t_l)\,\|B\|\left(M+\|u\|_{L^\infty}\right)\tilde{M} \\ &\le \left\|\int_s^{t_l} [u_n(\tau)-u(\tau)]\,e^{(t_l-\tau)A}B\Phi_u(\tau,s)x\,d\tau\right\|_H + \frac{\epsilon}{2}.\end{aligned}$$

As stated in the introduction, a sequence u_n converges weakly* to u if and only if for any $f \in L^1([s,T];H)$

$$\lim_{n\to+\infty}\left\|\int_s^T [u_n(\tau)-u(\tau)]\,f(\tau)\,d\tau\right\|_H = 0.$$

It is easy to see that the functions in the last integral

$$f_l(\tau) = \begin{cases} e^{(t_l-\tau)A}B\Phi_u(\tau,s)x; & \tau\in[s,t_l], \\ 0; & \text{elsewhere}, \end{cases}$$

are all in $L^1([s,T];H)$ for $l = 0,1,\dots,m-1$. There is a sufficiently large N such that for all $n \geq N$ and any t_l, $l = 0,1,\dots,m-1$,

$$\left\| \int_s^T [u_n(\tau) - u(\tau)]\, f_l(\tau)\, d\tau \right\|_H < \frac{\epsilon}{2}.$$

This means that for all $n \geq N$

$$\left\| \int_s^t [u_n(\tau) - u(\tau)]\, e^{(t-\tau)A} B \Phi_u(\tau, s) x\, d\tau \right\| < \epsilon.$$

Applying again Gronwall's inequality with this fact to the above differential inequality (8) leads to

$$\|\Phi_{u_n}(t,s)x - \Phi_u(t,s)x\|_H < \epsilon\ \exp(TM\|B\|)$$

for all $t \in [s,T]$ and all $n \geq N$, or

$$\sup_{t\in[s,T]} \|\Phi_{u_n}(t,s)x - \Phi_u(t,s)x\|_H \leq \epsilon\ \exp(TM\|B\|)$$

for all $n \geq N$. Equivalently, we can say that

$$\lim_{n\to+\infty} \|\Phi_{u_n}(\cdot\,,s)x - \Phi_u(\cdot\,,s)x\|_{\mathcal{C}([s,T];H)} = 0. \qquad \blacksquare$$

For each $T > 0$, $0 \leq s \leq t \leq T$, and $x \in H$, we define the application

$$\Gamma_{s,x}\ : L^\infty[s,T] \longrightarrow \mathcal{C}([s,T];H), \quad \Gamma_{s,x}(u)(t) = \Phi_u^*(t,s)x.$$

Then the following result can be proved in a similar way as for Lemma 1.

LEMMA 2. *For each $T > 0$, $0 \leq s \leq T$, and $x \in H$ the application $\Gamma_{s,x}$ is continuous with respect to the weak* topology on $L^\infty[s,T]$ and the uniform topology on $\mathcal{C}([s,T];H)$. For each $0 \leq s \leq t < +\infty$ and a positive element $u \in L^\infty(\mathbb{R}^+)$, the evolution operator $\Phi_u(t,s)$ is a contraction.*

PROOF. Let us prove the variation of constant formula,

$$\Phi_u(t,s)x = e^{(t-s)A}x + \int_s^t \Phi_u(t,\tau)u(\tau)Be^{(\tau-s)A}x\, d\tau. \tag{9}$$

It is important to note that for each $x \in \mathcal{D}(A)$

$$\frac{\partial}{\partial\alpha}\Phi_u(t,\alpha)x = -\Phi_u(t,\alpha)(A + Bu(\alpha))x \tag{10}$$

a.e. for $s \leq \alpha \leq t \leq T$ (see [5, p. 42]). For $x \in \mathcal{D}(A)$, we can differentiate the function $g(\alpha) = \Phi_u(t,\alpha)e^{(\alpha-s)A}x$ with respect to α for $x \in \mathcal{D}(A)$, and then integrate it from s to t in order to prove the formula. The domain $\mathcal{D}(A)$ being dense in H, the formula is true for all $x \in H$. Then it follows from the identity (9) that for all $x \in H$,

$$\Phi_u^*(t,s)x = e^{(t-s)A^*}x + \int_s^t u(\tau)e^{(\tau-s)A^*}B^*\Phi_u^*(t,\tau)x\, d\tau.$$

By assumption, we know that the operator A^* is dissipative [6, p. 41] and the operator B^* is bounded. Then exactly the same technique as that in the proof of Lemma 1 can be used to prove the first part of Lemma 2.

Since the set of piecewise constant functions is dense in $L^\infty[s,t]$ with respect to the weak* topology, we can divide the interval $[s,t]$ into n subintervals $[t_i, t_{i+1}]$ such that $t_0 = s$ and $t_n = t$. Consider the sequence of functions u_n which is equal to the nonnegative constant r_i (which is possible[1]) on $[t_i, t_{i+1}]$, $i = 0, 1, \dots, n-1$ and which converges weakly* to u in $L^\infty[s,t]$. It is clear that for all $x \in H$,

$$\Phi_{u_n}(t,s)x = \exp\left((t_n - t_{n-1})(A + r_{n-1}B)\right) \circ \cdots \circ \exp\left((t_1 - t_0)(A + r_0 B)\right)x. \tag{11}$$

Since the operator $A + r_i B$ is dissipative,

$$\|\Phi_{u_n}(t,s)x\| \le \|x\|.$$

Taking the limit for $n \to +\infty$ and using the Lemma 1, we prove that for all $x \in H$,

$$\|\Phi_u(t,s)x\| \le \|x\|.$$

Hence, the Lemma 2 is proved. ∎

From Lemma 2, we can see that for all $\varepsilon_0 \in H$ and all positive function $u \in L^\infty(\mathbb{R}^+)$, the mild solution of the estimation error $\varepsilon(t) = \Psi_u(t,s)\varepsilon_0$ satisfies the inequality

$$\|\Psi_u(t,s)\varepsilon_0\| \le \|\varepsilon_0\|$$

for all $s \le t < +\infty$. In other words, the positive function $\|\Psi_u(t,s)\varepsilon_0\|$ is a decreasing function of time t. Therefore, the limit $\lim_{t\to+\infty} \|\Psi_u(t,s)\varepsilon_0\|$ exists.

We have the following result.

LEMMA 3. *For each positive input $u \in L^\infty(\mathbb{R}^+)$, $\varepsilon \in H$ and s, $\alpha \ge 0$, we have*

$$\lim_{t\to+\infty} \int_0^\alpha |\langle \Psi_u(t+\tau,s)\varepsilon, c\rangle|^2 \, d\tau = 0. \tag{12}$$

PROOF. Notice that it is sufficient to prove the identity (12) for each $\varepsilon \in \mathcal{D}(A)$. Then the identity is true for each $\varepsilon \in H$ by the denseness of $\mathcal{D}(A)$ in H and the uniform boundedness of $\Psi_u(t,s)$.

For each $0 \le s \le t < +\infty$ and $\alpha \ge 0$, the input signal $u \in L^\infty[s, t+\alpha]$. From the proof of Lemma 2, we know that there is a sequence of piecewise constant positive functions u_n which converges weakly* to u in $L^\infty[s,t+\alpha]$ and that the evolution operator $\Psi_{u_n}(t,s)$ maps $\mathcal{D}(A)$ into $\mathcal{D}(A)$, (see the expression (11)). Hence, for each $\varepsilon \in \mathcal{D}(A)$, we can differentiate (except a finite number of points) with respect to t the function $(1/2)\|\Psi_{u_n}(t,s)\varepsilon\|^2$ following the error equation (4). Direct computations lead to

$$\begin{aligned}
\frac{d}{2dt}\|\Psi_{u_n}(t,s)\varepsilon\|^2 &= \langle A\Psi_{u_n}(t,s)\varepsilon, \Psi_{u_n}(t,s)\varepsilon\rangle \\
&\quad + u_n \langle B\Psi_{u_n}(t,s)\varepsilon, \Psi_{u_n}(t,s)\varepsilon\rangle - |\langle \Psi_{u_n}(t,s)\varepsilon, c\rangle|^2 \\
&\le -|\langle \Psi_{u_n}(t,s)\varepsilon, c\rangle|^2
\end{aligned}$$

a.e., since the operators A and B are dissipative. It follows that

$$\int_0^\alpha |\langle \Psi_{u_n}(t+\tau,s)\varepsilon, c\rangle|^2 \, d\tau \le \frac{1}{2}\|\Psi_{u_n}(t,s)\varepsilon\|^2 - \frac{1}{2}\|\Psi_{u_n}(t+\alpha,s)\varepsilon\|^2.$$

[1] It is sufficient to take a bounded sequence of continuous functions $u_k(t)$ which converges a.e. to $u(t)$. Then it is obvious that the bounded sequence of positive continuous function $|u_k(t)|$ converges a.e. to $u(t)$ (see Lusin's Theorem, [9, p. 53]).

From Lemma 1, by taking the limit for $n \to +\infty$, we get

$$\int_0^\alpha |\langle \Psi_u(t+\tau,s)\varepsilon, c\rangle|^2 \, d\tau \leq \frac{1}{2} \|\Psi_u(t,s)\varepsilon\|^2 - \frac{1}{2} \|\Psi_u(t+\alpha,s)\varepsilon\|^2 .$$

Taking the limit for $t \to +\infty$ in the above inequality we obtained the required result (12). ∎

Roughly speaking, we would like to deduce using the observability assumption that the estimation error tends to zero if the output does. However, the reasoning is not that simple. The next technical lemma is easy to prove from the definition of (5) and the first part of Lemma 2.

LEMMA 4. *The map from $L^\infty[0,T]$ equipped with the weak* topology to $\mathcal{L}(H,H)$ the Banach space of bounded linear operators on H:*

$$u \longrightarrow W(u,T)$$

is continuous for any $T > 0$.

LEMMA 5. *For each $T > 0$ and any positive input $u \in L^\infty(\mathbb{R}^+)$, we have*

$$\left\|W\left(u_{[t]},T\right)\varepsilon(t)\right\|_H \overset{t\to+\infty}{\longrightarrow} 0.$$

PROOF. The following variation of constant formula can be proved in the same way as (9):

$$\Phi_u(t+\xi,t)\varepsilon(t) = \varepsilon(t+\xi) + \int_t^{t+\xi} \Phi_u(t+\xi,\eta)c\,\langle\varepsilon(\eta),c\rangle \, d\eta. \tag{13}$$

The uniqueness of the mild solutions says that

$$\Phi_{u_{[t]}}(\xi,0) = \Phi_u(t+\xi,t). \tag{14}$$

We can write

$$\begin{aligned} W\left(u_{[t]},T\right)\varepsilon(t) = &\int_0^T \langle\varepsilon(t+\xi),c\rangle\, \Phi^*_{u_{[t]}}(\xi,0)c \, d\xi \\ &+ \int_0^T \int_0^\xi \langle\Phi_u(t+\xi,t+\eta)c,c\rangle \, \langle\varepsilon(t+\eta),c\rangle \, \Phi^*_{u_{[t]}}(\xi,0)c \, d\eta \, d\xi. \end{aligned}$$

The evolution operator $\Phi_u(t,s)$ being contractive as well as its adjoint, it is easy to see that

$$\begin{aligned} \left\|W\left(u_{[t]},T\right)\varepsilon(t)\right\| &\leq \int_0^T |\langle\varepsilon(t+\xi),c\rangle| \, d\xi \|c\| + \int_0^T |\langle\varepsilon(t+\eta),c\rangle| \, d\eta \|c\|^3 T \\ &\leq \|c\|(1+T\|c\|^2)\sqrt{T}\left[\int_0^T |\langle\varepsilon(t+\xi),c\rangle|^2 \, d\xi\right]^{1/2}. \end{aligned}$$

Applying Lemma 3 gives the required result,

$$\lim_{t\to+\infty} \left\|W\left(u_{[t]},T\right)\varepsilon(t)\right\|_H = 0.$$

∎

LEMMA 6. *Given a positive input $u \in L^\infty(\mathbb{R}^+)$ and a sequence θ_n such that $u_{[\theta_n]}$ converges weakly* to a T-universal input $u^* \in L^\infty[0,T]$ when $\theta_n \overset{n\to+\infty}{\longrightarrow} +\infty$, then $\varepsilon(\theta_n)$ tends weakly to zero in H.*

PROOF. From Lemma 4 and Lemma 5, we see that

$$\lim_{\theta_n \to +\infty} \|W(u^*, T)\, \varepsilon(\theta_n)\|_H = 0. \tag{15}$$

Since $\varepsilon(\theta_n)$ is bounded, it contains weakly convergent subsequences. Let us pick an arbitrary subsequence $\varepsilon(\theta_{nk})$ converging weakly to ξ. The sequence $\varepsilon(\theta_n)$ converges weakly to zero if and only if every weakly converging subsequence does. Let us prove that $\xi = 0$.

Since the Gram-observability operator $W(u,T)$ for $u \in L^\infty[0,T]$ is a compact operator, $\varepsilon(\theta_{nk})$ converging weakly to ξ implies that $W(u^*,T)\varepsilon(\theta_{nk})$ converges strongly to $W(u^*,T)\xi$. It follows from the above (15) that

$$W(u^*, T)\, \xi = 0.$$

The input u^* being T-universal, we have $\xi = 0$. ∎

LEMMA 7. *Given a positive input $u \in L^\infty(\mathbb{R}^+)$, a sequence θ_n such that $\theta_n \overset{n\to+\infty}{\longrightarrow} +\infty$ and a bounded sequence $0 \le \tau_n \le \alpha < +\infty$, if $u_{[\theta_n]}$ converges weakly* to a T-universal input $u^* \in L^\infty[0,T]$ for some $T > 0$ when $n \to +\infty$, then*

$$\varepsilon(\theta_n + \tau_n) \overset{n\to+\infty}{\longrightarrow} 0 \text{ weakly in } H.$$

PROOF. A similar argument of the proof of Lemma 5 can be used to prove that the difference

$$\Phi_{u_{[\theta_n]}}(\tau_n, 0)\, \varepsilon(\theta_n) - \varepsilon(\theta_n + \tau_n)$$

converges strongly to zero as $n \to +\infty$. Indeed, using the fact that the evolution operator is contractive, we have

$$\left\|\Phi_{u_{[\theta_n]}}(\tau_n, 0)\, \varepsilon(\theta_n) - \varepsilon(\theta_n + \tau_n)\right\| \le \sqrt{\alpha}\left[\int_0^\alpha |\langle \varepsilon(\theta_n + \eta), c\rangle|^2\, d\eta\right]^{1/2} \|c\|.$$

Moreover, by mean of Lemma 2 and Lemma 6, the first term of the difference converges weakly to zero: for all $y \in H$,

$$\lim_{n\to+\infty} \langle \Phi_{u_{[\theta_n]}}(\tau_n, 0)\, \varepsilon(\theta_n), y\rangle = \lim_{n\to+\infty} \left\langle \varepsilon(\theta_n), \Phi^*_{u_{[\theta_n]}}(\tau_n, 0)\, y\right\rangle = 0. \tag{16}$$

Therefore, the second term $\varepsilon(\theta_n + \tau_n)$ converges also weakly to zero as required. ∎

2.1. Proof of Theorem 1

Since u is a regularly persistent input, there exists a sequence θ_n, with $\theta_{n+1} - \theta_n$ bounded, for which $u_{[\theta_n]}$ verifies the conditions of Lemma 7. Then, for any sequence r_n such that $r_n \overset{n\to+\infty}{\longrightarrow} +\infty$, we can find a bounded sequence τ_n such that $r_n = \theta_{k(n)} + \tau_n$, where $\theta_{k(n)}$ denotes a subsequence of θ_n. Applying Lemma 7, we get directly $\lim_{n\to+\infty} \varepsilon(r_n) = 0$ weakly in H, equivalently, $\varepsilon(t) \overset{t\to+\infty}{\longrightarrow} 0$ weakly in H. ∎

2.2. Proof of Corollary 1

If we prove that for every bounded sequence $0 \le \tau_n \le T$ and every $u \in L^\infty(\mathbb{R}^+)$, $\varepsilon(nT + \tau_n)$ $\overset{n\to+\infty}{\longrightarrow} 0$ weakly in H (where $n = 1, 2\dots$), then the same idea as that in the proof of Theorem 1 can be applied to prove Corollary 1.

Now take an arbitrary weakly converging subsequence $\varepsilon(n_kT + \tau_{n_k})$ of the sequence $\varepsilon(nT + \tau_n)$ such that for all $x \in H$,

$$\lim_{k\to+\infty} \langle \varepsilon(n_kT + \tau_{n_k}), x\rangle = \langle \xi^*, x\rangle.$$

In particular, we have:

$$\lim_{k\to+\infty} \langle \varepsilon\left(n_k T + \tau_{n_k}\right), \xi^* \rangle = \|\xi^*\|^2 .$$

Let us prove that $\xi^* = 0$. Assume that $\xi^* \neq 0$. Extract further from $u_{[n_k T]}$ a weakly* converging subsequence $u_{[n_{k_l} T]}$ such that $u_{[n_{k_l} T]} \overset{l\to+\infty}{\longrightarrow} u^*$ weakly* which is T-universal by assumption. It follows from Lemma 7 that

$$\lim_{l\to+\infty} \left\langle \varepsilon\left(n_{k_l} T + \tau_{n_{k_l}}\right), \xi^* \right\rangle = \|\xi^*\|^2 = 0.$$

This contradiction proves that $\xi^* = 0$. Hence, the whole sequence $\varepsilon(nT + \tau_n)$ converges weakly to zero, and this proves Corollary 1. ∎

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